

# Causal transformation of Gödel-type space-times in conformal field theory

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September 27, 2010

## Abstract

The Gödel-type metrics are considered as backgrounds of the sigma-models. In the conformal field theory such backgrounds are deformed by the exactly marginal operators. We examine, how the closed time-like curves (CTC's) transform under such deformations.

**PACS** :11.25.Hf ; 04.20.Gz

**Keywords** : Gödel-type metrics, marginal deformations.

## 1 Introduction

Space-times with the closed timelike curves (CTC) are not rare solutions in general relativity. Some parts of these solutions are non-singular. However they are not considered as physical because they violate causality principle and lead to the well-known time paradoxes. From the other side one can suspect that these solutions should have counterparts in quantum theory of gravitation. This quantum theory should give a selection rule which forbids space-times with CTC on the classical level. Hawking called this rule chronology protection [1]. If such selection rule exists, then time traveling in the past is impossible. In other case a time machine is possible. Since theory of the full quantum gravity does not exist one can only study approximately this problem. String/M-theory is such approximation of the quantum gravity. Thus it seems interesting to examine conditions under which one can embed space-times with CTC into string/M-theory. In the supergravity approximation of string theory/M-theory such embeddings were considered [2-5]. Also various mechanisms which protect from CTC were considered [2,6]. In the case of the Gödel type space-times such rule were proposed in [2] with the use of the holography.

The Gödel space-time is an example of a homogenous solution with closed time-like curves through every point. It is also non-singular with a globally well-defined timelike Killing vector. In [3] the supersymmetric extensions of the Gödel space-time in five dimensions has been obtained by using Killing spinors

(in the supersymmetric case the Killing vectors are obtained from covariantly constant Killing spinors). Another way of obtaining Gödel space-times in string theory/M-theory is the T or S-dualizing a supersymmetric pp-wave solutions [2,4].

The Gödel space-time has been obtained as an exact solution  $M_3$  of three-dimensional gravity coupled to a Maxwell-Chern-Simons (MCS) theory [7]. It is also known that the four-dimensional Gödel space-time has the structure  $M_3 \times \mathbf{R}^1$ . The gauge group of MCS is  $U(1)$ . As turns out the 3-dimensional gauge theories with Chern-Simons (CS) terms are realized in brane constructions in type IIB string theory [8,9]. In [10] was shown that the 3-dimensional  $\mathcal{N} = 6$  supersymmetric  $U(N) \times U(N)$  Chern-Simons theory coupled to the matter is equivalent to the low-energetic theory on  $N$  M2-branes at a  $\mathbf{C}^4/\mathbf{Z}_k$  singularity and  $k$  is a level of the Chern-Simons theory. This observation follows from the fact that the moduli space of the supersymmetric Chern-Simons is  $\mathbf{C}^4/\mathbf{Z}_k$  which is the same as the moduli space for  $N$  M2-branes probing  $\mathbf{C}^4/\mathbf{Z}_k$  singularity in M-theory. For  $N \gg 1$  this low-energetic theory has a dual description in terms of M-theory on the background  $AdS_4 \times S^7/\mathbf{Z}_k$ . For  $N/k$  fixed and  $N \rightarrow \infty$  (the t'Hooft limit) the dual theory becomes type IIA string theory on  $AdS_4 \times \mathbf{CP}^3$  background. From the other side the 3-dimensional Gödel space-time is obtained as the result of the wrapping of M2-branes in the flux compactification in M-theory [11].

As turns out Gödel and AdS metrics in (2+1) dimensions belong to the one parameter family metrics [12]. In this family the Anti de Sitter (AdS) is the boundary between space-times with CTC and without CTC. Thus it is interesting to relate Gödel-type solutions to AdS/CFT duality, in particular, what closed timelike curves (CTC) mean in conformal field theory (CFT). Some proposal relating causality region (without CTC) in the case of the rotating black hole to unitarity of the dual CFT is given in [13]. The AdS/CFT duality relate the string theory on 10-dimensional space-time  $X$  (which is asymptotic to  $AdS_n \times K_{10-n}$ ) to CFT defined on conformal boundary of  $AdS_n$ . This duality is now well established and was checked for different  $n$ . One can ask what will be changed in AdS/CFT duality if AdS is replaced by one parameter family metrics of [12]. The heterotic string propagating on deformed 3-dimensional AdS spaces were considered in [14]. These backgrounds correspond to exactly marginal deformations of the worldsheet conformal field theory. The string theory models in Taub-NUT geometry with CTC's were considered in [15].

From the above one can see that the problem on causality is translated on language of conformal field theory and/or string theory.

In this paper we consider Gödel-type metrics in three dimensions from the point of views of gauge theory (in section 2) and sigma model (in section 3). We will be interested in conditions of appearing CTC's and their behavior in the sigma models. Because gravity in 2+1 dimensions is expressed in terms of gauge theory [16] so the gravity and MCS become the coupled system of the two gauge theories: the first one is the non-abelian corresponding to gravity and the second one is abelian corresponding to Maxwell field. Thus CTC should have counterparts in such coupled gauge theory. In section 2 we recall properties

of gravity in (2+1)-dimensions as the Chern-Simons theory and properties of the Gödel-type metrics. In this section we also give relation between coupling constants of these gauge theories in the case of the CTC. In section 3 we consider sigma-model with target given by Gödel-type space-time and show that exist transformations which transform space-time with CTC on space-time without CTC. Section 4 is devoted to conclusions. In Appendix we recall the symmetry of Gödel-type metrics and relation with AdS metric.

## 2 Gödel metrics and gauge theories in 3-dimensions

The Hilbert-Einstein action with a cosmological constant  $\Lambda$  in (2+1)-dimensional space-time  $M$  has the form:

$$S[g_{\alpha\beta}] = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} (R - 2\Lambda), \quad (2.1)$$

where  $g_{\alpha\beta}$  is a metric on  $M$ . As is well-known one can go from the metric  $g_{\alpha\beta}$  to the first order forms  $e^a = e^a_\alpha dx^\alpha$  and spin connections  $\omega^a = \omega^a_\alpha dx^\alpha$  and as the result obtains the action  $S$  in terms of  $e$  and  $\omega$ :

$$S[e, \omega] = \frac{1}{16\pi G} \int_M \left[ e^a \wedge \left( d\omega^a + \frac{1}{2} \varepsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{\Lambda}{6} \varepsilon_{abc} e^a \wedge e^b \wedge e^c \right]. \quad (2.2)$$

The above action can be written as a sum of the Chern-Simons (CS) terms with  $SL(2, \mathbf{R})$  gauge groups only in (2+1)-dimensions [16,17]. For negative cosmological constant  $\Lambda = -1/l^2$  the action takes the form:

$$S[A_L, A_R] = k_L CS[A_L] - k_R CS[A_R], \quad (2.3)$$

where  $CS[A]$  is the Chern-Simons term:

$$CS[A] = \frac{1}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (2.4)$$

and  $k_L + k_R = l/(8G)$ . The one-forms  $A_{L,R}^a = \omega^a \pm l e^a$  are independent with values in  $sl(2, \mathbf{R})$ . In this way on the classical level Einstein gravity in (2+1) space-time is the same as the CS theory with the gauge group  $SL(2, \mathbf{R}) \times SL(2, \mathbf{R}) \approx SO(2, 2)$ . This gauge group is the symmetry of the 3-dimensional AdS space-time. For this gauge theory one can add abelian gauge field  $b_\alpha$  with Maxwell action and abelian Chern-Simons term. The action for this system is:

$$S_{MCS}[a] = \frac{1}{16\pi G} \int_M d^3x \left( \frac{\sqrt{-g}}{4} F^{\alpha\beta} F_{\beta\alpha} - \frac{k_A}{4\pi} \varepsilon^{\alpha\beta\gamma} b_\alpha F_{\beta\gamma} \right), \quad (2.5)$$

where  $F_{\alpha\beta} = \partial_\alpha b_\beta - \partial_\beta b_\alpha$ . Thus the gravity and MCS theory is the coupled gauge theory with the gauge group  $SO(2, 2) \times SO(2)$  and the action:

$$S[A_L, A_R, b] = k_L CS[A_L] - k_R CS[A_R] - k_A CS[b] + S[A_{L,R}, db], \quad (2.6)$$

where the last term is interaction part of the theory and is exactly the Maxwell action for  $b$ :

$$S[A_{L,R}, db] = \frac{1}{4} \int db \wedge *db. \quad (2.7)$$

The Hodge star  $*$  is expressed in terms of connections  $A_L$  and  $A_R$ . As was shown in [7] such the system has as a solution in the form of the Gödel-type metric and the Maxwell field  $F = db$ . This metric and the gauge abelian field  $b$  can be written as follows [18]:

$$ds^2 = - \left( dt + \frac{4\Omega}{m^2} \sinh^2 \left( \frac{m\rho}{2} \right) d\phi \right)^2 + d\rho^2 + \frac{1}{m^2} \sinh^2(m\rho) d\phi^2, \quad (2.8)$$

$$b(\rho) = \frac{4}{m^2} (\Omega^2 - 1/l^2)^{1/2} \sinh^2 \left( \frac{m\rho}{2} \right) d\phi, \quad (2.9)$$

where the cosmological constant  $1/l$  is related to  $\Omega$  and  $m$  as follows:  $m^2 - 2\Omega^2 = 2/l^2$ . The vorticity  $\Omega$  is related with the Chern-Simons coupling  $k_A$ :  $\Omega = k_A/(2\pi)$ . Then we obtain relation between coupling constants and the parameter  $m$ :

$$m^2 = \frac{k_A^2}{2\pi^2} + \frac{2}{(8G)^2 (k_L + k_R)^2}. \quad (2.10)$$

From the topological reasons the coupling constants  $(k_R - k_L)$  and  $k_A$  are integers [17]. The metric (2.8) and the field (2.9) have the following limits when  $m \rightarrow 0$ :

$$ds^2 = - (dt + \Omega \rho^2 d\phi)^2 + d\rho^2 + \rho^2 d\phi^2, \quad (2.11)$$

$$b(\rho) = (\Omega^2 - 1/l^2)^{1/2} \rho^2 d\phi. \quad (2.12)$$

In this limit CTC's appear if  $\rho > 1/\Omega$ .

The metric (2.8) can also be expressed as the one parameter family of the 3-dimensional metrics interpolating between the Gödel metric and the AdS metric [12]:

$$ds_\mu^2 = 4a^2 \{ -d\tau^2 + dr^2 + \sinh^2(r) [1 + (1 - \mu^2) \sinh^2(r)] d\phi^2 - 2\mu \sinh^2(r) d\tau d\phi \}, \quad (2.13)$$

where the parameter  $\mu$  and the radius  $a$  are related to vorticity  $\Omega$  and  $m$  in the following way:  $\mu^2 = 4\Omega^2/m^2$  and  $a^2 = 1/m^2$ . The coordinates  $\tau$  and  $r$  are related to  $t$  and  $\rho$  by transformation:  $t = 2\tau/m$  and  $\rho = 2r/m$ . The CTC appears if  $g_{\phi\phi}(r)$  becomes the time-like:  $g_{\phi\phi}(r) < 0$  which occurs for  $r > r_c = \frac{1}{2} \ln \left( \frac{\mu+1}{\mu-1} \right)$  and  $\mu > 1$ . As it is well-known the AdS metric and the Gödel metric are obtained for  $\mu^2 = 1$  and for  $\mu^2 = 2$  respectively. Using (2.10) the parameter  $\mu^2$  is equal to:

$$\mu^2 = \frac{2k_A^2}{k_A^2 + (\pi/4G)^2 (k_L + k_R)^{-2}}. \quad (2.14)$$

Thus the CTC appears if:

$$k_A^2 (k_L + k_R)^2 > (\pi/4G)^2. \quad (2.15)$$

As one can see under the change of  $k_A$  on  $k_L + k_R$  the eq. (2.14) and the condition (2.15) are invariant.

The AdS metric is obtained if:  $k_A^2 (k_L + k_R)^2 = (\pi/4G)^2$ . In the case of the Gödel metric ( $\mu^2 = 2$ ) one gets that  $(k_L + k_R)^{-2} = 0$ . It means that  $\Lambda = 0$  and  $\Lambda$  is replaced by  $\Omega$  given by  $k_A$ . In this way we obtained condition on CTC expressed by the coupling constants of the gauge theory. It is similar to the condition on BPS states. The state is BPS if a mass and a charge of the state are equal. In our case the CTC disappears if (2.15) becomes equality what corresponds to the AdS metric.

### 3 Sigma-model in the Gödel type backgrounds

In the conformal gauge on the worldsheet  $\Sigma_2$  the bosonic part of the action for sigma-model has the form:

$$S = \frac{1}{2\pi} \int_{\Sigma_2} d^2z (G_{MN} + B_{MN}) \partial X^M \bar{\partial} X^N + \frac{1}{4\pi} \int_{\Sigma_2} \Phi(X) R^{(2)} d^2z, \quad (3.1)$$

where  $X$  is the mapping from  $\Sigma_2$  to the manifold  $M_n$  with the background metric  $G_{MN}$ , antisymmetric form  $B_{MN}$  and dilaton field  $\Phi$ . The scalar curvature of  $\Sigma_2$  is denoted by  $R^{(2)}$ . In the case when the background is given by the metric (2.8) and the field  $X$  is parametrized as follows:  $X = (\tau, r, \phi)$ , then the action takes the form:

$$S[\tau, r, \phi] = \frac{4a^2}{2\pi} \int_{\Sigma_2} d^2z [-\partial\tau\bar{\partial}\tau + \partial r\bar{\partial}r + (\sinh^2(r) + (1 - \mu^2) \sinh^4(r)) \partial\phi\bar{\partial}\phi - \mu \sinh^2(r) (\partial\phi\bar{\partial}\tau + \bar{\partial}\phi\partial\tau)] + \frac{1}{4\pi} \int_{\Sigma_2} \Phi R^{(2)} d^2z, \quad (3.2)$$

with the background symmetry generated by the algebra  $so(2) \times sl(2, \mathbf{R})$ . If one considers the above background in the sigma model, then the consistency conditions need a non-zero B-field in order to solve the sigma-model beta function equations. However let us assume here that the field B is equal to zero. As one can notice this action is invariant under shift symmetry in the variables  $\tau$

and  $\phi$ . Since this symmetry is abelian we can rewrite the action as follows [19]:

$$S[\tau, r, \phi] = \frac{1}{2\pi} \int_{\Sigma_2} d^2z \begin{pmatrix} \partial\tau & \partial\phi & \partial r \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & 4a^2 \end{pmatrix} \begin{pmatrix} \bar{\partial}\tau \\ \bar{\partial}\phi \\ \bar{\partial}r \end{pmatrix} + \frac{1}{4\pi} \int_{\Sigma_2} \Phi R^{(2)} d^2z, \quad (3.3)$$

where the matrix  $E$  is equal to:

$$E = 4a^2 \begin{pmatrix} -1 & -\mu \sinh^2 r \\ -\mu \sinh^2 r & u(r) \sinh^2 r \end{pmatrix} \quad (3.4)$$

and  $u(r) = 1 + (1 - \mu^2) \sinh^2 r$ . We can deform this conformal field theory by exactly marginal operators. In the case of the toroidal background these operators correspond to a one parameter families of  $O(d, d, \mathbf{R})$  rotations, where  $d$  is a number of the abelian isometries. In the considered model  $d = 2$  thus on the target space side the exactly marginal operators correspond to a one parameter families of  $O(2, 2, \mathbf{R})$  rotations. The group  $O(2, 2, \mathbf{R})$  acts on the background represented by a matrix  $E = G + B$  in the following way:

$$g(E) = (aE + b)(cE + d)^{-1}, \quad (3.5)$$

where  $g \in O(2, 2, \mathbf{R})$  has the matrix representation:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.6)$$

and the matrices  $a, b, c$  and  $d$  fulfill relations:  $a^T d + c^T a = b^T d + d^T b = 0$ ,  $a^T d + c^T b = I_2$ . The maximal compact subgroup of  $O(2, 2, \mathbf{R})$  is  $O(2) \times O(2)$ . The embedding  $e : O(2) \times O(2) \rightarrow O(2, 2, \mathbf{R})$  has the form:

$$e(r_1, r_2) = \frac{1}{2} \begin{pmatrix} r_1 + r_2 & r_1 - r_2 \\ r_1 - r_2 & r_1 + r_2 \end{pmatrix}, \quad (3.7)$$

where  $(r_1, r_2) \in O(2) \times O(2)$ . This subgroup depends on two angles  $\alpha_1$  and  $\alpha_2$  which parametrize the element  $(r_1(\alpha_1), r_2(\alpha_2))$  in the standard way:

$$r_i(\alpha_i) = \begin{pmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{pmatrix} \quad (3.8)$$

and  $i = 1, 2$ . Thus the embedding  $e$  has the form:

$$e(r_1, r_2) = \begin{pmatrix} r(\alpha) \cos \beta & r(\alpha) \varepsilon \sin \beta \\ r(\alpha) \varepsilon \sin \beta & r(\alpha) \cos \beta \end{pmatrix} \equiv e_{(\alpha, \beta)}, \quad (3.9)$$

where the angles  $\alpha$  and  $\beta$  are related to  $\alpha_1$  and  $\alpha_2$  as follows:  $\alpha = (\alpha_1 + \alpha_2)/2$ ,  $\beta = (\alpha_1 - \alpha_2)/2$  the matrices  $r(\alpha)$  and  $\varepsilon$  are:

$$r(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \text{ and } \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.10)$$

The action of this maximal compact group on  $E$  depends on two parameters  $\alpha$  and  $\beta$  and gives new background matrix  $\tilde{E}$ :

$$e_{(\alpha, \beta)}(E) \equiv \tilde{E} = r(\alpha) [E + \varepsilon \tan \beta] [\varepsilon E \tan \beta + I]^{-1} r^T(\alpha). \quad (3.11)$$

In the considered case the background field  $B$  is vanishing thus the matrix  $E = G$  and the metric  $G$  is given by (2.13). Under the deformations related to the maximal compact subgroup  $O(2) \times O(2)$  the matrix  $\tilde{E}$  has the entries:

$$\begin{aligned} \tilde{E}_{11} = & W_\beta(r) [(1 - \mu^2) \sin^2 \alpha \sinh^4 r \\ & + (\sin^2 \alpha - \mu \sin 2\alpha) \sinh^2 r - \cos^2 \alpha], \end{aligned} \quad (3.12)$$

$$\begin{aligned} \tilde{E}_{12} = & \frac{W_\beta(r)}{2} [(1 - \mu^2) \sin 2\alpha \sinh^4 r \\ & + (\sin 2\alpha - 2\mu \cos 2\alpha) \sinh^2 r + \sin 2\alpha + 2\tilde{b}], \end{aligned} \quad (3.13)$$

$$\begin{aligned} \tilde{E}_{21} = & \frac{W_\beta(r)}{2} [(1 - \mu^2) \sin 2\alpha \sinh^4 r \\ & + (\sin 2\alpha - 2\mu \cos 2\alpha) \sinh^2 r + \sin 2\alpha - 2\tilde{b}], \end{aligned} \quad (3.14)$$

$$\begin{aligned} \tilde{E}_{22} = & W_\beta(r) [(1 - \mu^2) \cos^2 \alpha \sinh^4 r \\ & + (\cos^2 \alpha - \mu \sin 2\alpha) \sinh^2 r - \sin^2 \alpha], \end{aligned} \quad (3.15)$$

where the functions  $W_\beta(r)$ ,  $\tilde{b}(r)$  are equal to:

$$\begin{aligned} W_\beta(r) = & \frac{4a^2 (1 + \tan^2 \beta)}{1 - 4a^4 \tan^2 \beta \sinh^2(2r)}, \\ \tilde{b} = & \frac{1}{8} (1 + 4a^2 \sinh^2(2r)) \sin(2\beta). \end{aligned} \quad (3.16)$$

One can read off from the above formulas that the new background has the metric  $\tilde{G}$  given by :

$$\tilde{G}_{11} = \tilde{E}_{11}, \quad \tilde{G}_{22} = \tilde{E}_{22} \quad (3.17)$$

and

$$\begin{aligned} \tilde{G}_{12}(\alpha, \beta) = & \frac{W_\beta(r)}{2} [(1 - \mu^2) \sin 2\alpha \sinh^4 r \\ & + (\sin 2\alpha - 2\mu \cos 2\alpha) \sinh^2 r + \sin 2\alpha], \end{aligned} \quad (3.18)$$

with the antisymmetric two form  $\tilde{B}$  :

$$\tilde{B} = \frac{W_\beta(r)}{8} (1 + 4a^2 \sinh^2(2r)) \sin(2\beta) d\tau \wedge d\phi. \quad (3.19)$$

For  $\alpha = \beta = 0$  we obtain the initial matrix  $E$ . The function  $W_\beta(r)$  is finite for  $\beta = \pi/2$  and has the value  $W_{\pi/2}(r) = -a^{-2} \sinh^{-2}(2r)$ . In the case when  $\mu^2 > 1$  the forms of the  $\tilde{G}_{11}$ ,  $\tilde{G}_{12}$  and  $\tilde{G}_{22}$  are:

$$\tilde{G}_{11}(\alpha, \beta) = (\mu^2 - 1) W_\beta(r) h(R) \sin^2 \alpha, \quad (3.20)$$

$$\tilde{G}_{12}(\alpha, \beta) = \frac{1}{2} (\mu^2 - 1) W_\beta(r) g(R) \sin 2\alpha, \quad (3.21)$$

$$\tilde{G}_{22}(\alpha, \beta) = (\mu^2 - 1) W_\beta(r) f(R) \cos^2 \alpha, \quad (3.22)$$

where:

$$h(R) = \left( \frac{1 - 2\mu \cot \alpha}{2(\mu^2 - 1)} \right)^2 - \frac{\cot^2 \alpha}{\mu^2 - 1} - \left( R - \frac{1 - 2\mu \cot \alpha}{2(\mu^2 - 1)} \right)^2, \quad (3.23)$$

$$g(R) = \left( \frac{1 - 2\mu \cot 2\alpha}{2(\mu^2 - 1)} \right)^2 - 1 - \left( R - \frac{1 - 2\mu \cot 2\alpha}{2(\mu^2 - 1)} \right)^2, \quad (3.24)$$

$$f(R) = \left( \frac{1 - 2\mu \tan \alpha}{2(\mu^2 - 1)} \right)^2 - \frac{\tan^2 \alpha}{\mu^2 - 1} - \left( R - \frac{1 - 2\mu \tan \alpha}{2(\mu^2 - 1)} \right)^2 \quad (3.25)$$

and  $R = \sinh^2 r$ . One can notice that for  $\alpha = \pi/2$  the metric components are following:

$$\tilde{G}_{11}(\pi/2, \beta) = \tilde{G}_{22}(0, \beta) = \frac{1}{4a^2} W_\beta(r) G_{22}, \quad (3.26)$$

$$\tilde{G}_{22}(\pi/2, \beta) = \tilde{G}_{11}(0, \beta) = \frac{1}{4a^2} W_\beta(r) G_{11}, \quad (3.27)$$

$$\tilde{G}_{12}(\pi/2, \beta) = -\frac{1}{4a^2} W_\beta(r) G_{12}. \quad (3.28)$$

It means that under  $\alpha$ -rotation (for  $\alpha = \pi/2$ ) the time and space coordinates are changing their positions. In such rotated background there is a nonvanishing two-form  $\tilde{B}$  which depends on the second rotation angle  $\beta$ . Moreover for  $\alpha = \pi/2$  and  $\beta = \pi/2$  the deformed background is conformal equivalent to the initial with



the changing time and space coordinates with the vanishing two-form and the conformal factor is equal to  $W_{\pi/2}(r)$ . One can also notice the following relation:

$$\tilde{G}_{11}(\alpha, \beta) + \tilde{G}_{22}(\alpha, \beta) = (\mu^2 - 1) W_\beta(r) p(R), \quad (3.29)$$

where

$$p(R) = -R^2 - \frac{2\mu \sin 2\alpha - 1}{\mu^2 - 1} R + \frac{1}{\mu^2 - 1}. \quad (3.30)$$

Since the group  $O(2) \times O(2)$  has two generators thus the exactly marginal operators correspond to two independent one-parametric families related to the rotation in  $\blacksquare$  angle and the rotation in  $\blacksquare$  angle, respectively. This independence of the two families exactly marginal operators is reflected in the factorized form of the transformed background (3.19-3.22) where the factors depend on  $\blacksquare$  and  $\blacksquare$ .

In the background given by (3.19)-(3.22) CTC appears if  $\tilde{G}_{22}$  becomes negative whereas  $\tilde{G}_{11}$  remains also negative. Thus CTC appears if:  $f(R) < 0$  and  $W_\beta(r) > 0$  or  $f(R) > 0$  and  $W_\beta(r) < 0$  under condition that  $\tilde{G}_{11} < 0$ . This last condition implicated that:  $W_\beta(r) < 0$  and  $h(R) > 0$  or  $W_\beta(r) > 0$  and  $h(R) < 0$ . The function  $W_\beta(r) > 0$  for  $\sinh^2(2r) < (\cot^2 \beta) / (4a^2)$ . It is easy to see that  $f(R) > 0$  if  $R \in (R_-^f, R_+^f) \equiv I^f$  where:

$$R_\pm^f = \frac{1}{2(\mu^2 - 1)} \left( 1 - 2\mu \tan \alpha \pm 2\sqrt{(\tan \alpha - \mu/2)^2 - (\mu^2 - 1)/4} \right) \quad (3.31)$$

and  $f(R_\pm^f) = 0$ . One can see that  $R_\pm^f$  are real if  $\tan \alpha \in (0, \nu_-/2] \cup [\nu_+/2, +\infty) \equiv D_f$  where  $\nu_\pm = \mu \pm \sqrt{\mu^2 - 1}$ . In the case when  $\tan \alpha \in (\nu_-/2, \nu_+/2)$  the function  $f(R)$  is always negative. Let us consider the case with  $\blacksquare = \blacksquare/2$ . This choice of  $\blacksquare$  is a matter of convenience and simplifies computations. For other values of  $\blacksquare$  the qualitative picture of our considerations does not change. Thus in our case  $W_{\pi/2}(r) < 0$  and CTC exists only when  $f(R) > 0$  and  $h(R) > 0$ . The last inequality holds for  $R \in (R_-^h, R_+^h) \equiv I^h$  where:

$$R_\pm^h = \frac{1}{2(\mu^2 - 1)} \left( 1 - 2\mu \cot \alpha \pm 2\sqrt{(\cot \alpha - \mu/2)^2 - (\mu^2 - 1)/4} \right) \quad (3.32)$$

and  $h(R_\pm^h) = 0$ . These real roots exist if  $\tan \alpha \in (0, 2\nu_-] \cup [2\nu_+, +\infty) \equiv D_h$ . For  $\tan \alpha \in (2\nu_-, 2\nu_+)$  the function  $h(R)$  is always negative. Thus the real roots  $R_\pm^f$  and  $R_\pm^h$  exist if  $\tan \alpha \in D_f \cap D_h = (0, \nu_-/2] \cup [\nu_+/2, 2\nu_-] \cup [2\nu_+, +\infty)$ .

It follows that the CTC's appear iff  $I^f \cap I^h \neq \emptyset$ . This relation is valid if exists such  $R_0 > 0$  that  $f(R_0) = h(R_0) > 0$ . As one can easily find that  $R_0 = -(\mu \sin 2\alpha)^{-1}$  and this radius has to be greater than zero so we get that  $\alpha \in (\pi/2, \pi)$ . The value of the function  $f$  in the point  $R_0$  is:

$$f(R_0) = \frac{1}{(\mu^2 - 1) \sin^2(2\alpha)} Z(\alpha), \quad (3.33)$$

where:

$$Z(\alpha) = 4(\mu^3 - 1)\cos^4\alpha - (\mu^2 - 1)\sin^4\alpha - (4\mu\sin^2\alpha - \cos^2\alpha)\sin 2\alpha + \mu\sin^2(2\alpha) - \cos^2\alpha \quad (3.34)$$

and  $\alpha \in (\pi/2, \pi)$ . The sign of the function  $Z$  depends on the value of  $\mu$ . As the example we consider the Gödel metric ( $\mu^2 = 2$ ). The function  $Z(\alpha)$  is negative for  $\alpha \in [1.57, 1.8]$  (in radian) and becomes positive for  $\alpha > 1.801$ . Another considered example is for  $\mu^2 = (1.5)^2$  in this case we get that  $Z(\alpha)$  is negative for  $\alpha \in [1.57, 1.67]$ . Thus the transformation (3.11) with  $\alpha$  from these intervals (corresponding to  $\mu^2 = 2$  and  $\mu^2 = (1.5)^2$ ) takes the targets with CTC to the targets where the CTC is vanishing.

It means that in the considered case one can make starting with the Gödel-type metrics the transformation belonging to  $O(2) \times O(2)$  which changes the causal structure. In our case other transformations from  $O(2, 2, \mathbf{R})/(O(2) \times O(2))$  exist which form the moduli space for the CFT. The group  $O(2, 2, \mathbf{R})$  (in general case  $O(d, d, \mathbf{R})$  group) has interpretation as the duality group, where the T-duality is realized by  $O(2, 2, \mathbf{Z})$  subgroup. In general case where the number of isometries is  $d$  the group  $O(2, 2, \mathbf{R})$  is replaced by  $O(d, d, \mathbf{R})$ . This group is the structure group of the generalized tangent bundle, which combines the tangent and cotangent bundles of a  $d$ -dimensional manifold in generalized geometry [20, 21]. This suggests the use of generalized geometry to describe causal structure.

## 4 Conclusions

We considered space-times with the closed time-like curves. These space-times were represented by the Gödel-type metrics in three dimensions. We interested in the conditions of appearing CTC's and their behavior from the point of views gauge theory and conformal theory. In the section 2 we obtained condition on CTC's expressed by the coupling constants of the gauge theory. In the section 3 we started with the sigma-model with the target space given by Gödel-type spacetimes. Assuming that this model realize the conformal field theory we used exactly marginal operators represented by  $O(2, 2)$  on the target space side in order to get new targets which are not connected by target space diffeomorphisms. This assumption is justified by the results that Gödel-type spacetimes can be considered as the backgrounds for string propagation [4, 11, 14]. We obtained that in some of these new targets CTC's are vanishing whereas in other they still exist. Thus the exactly marginal operators act on the causal structure of the target. The transformed backgrounds have been here shown in the explicit forms (3.17-3.19). One can then see the analogy of our result with the relation between the mirror symmetry and the  $N = 2$  superconformal field theory realized by the sigma-model with the Calabi-Yau three-fold as a target. Note that the exactly marginal operators in the superconformal field theory correspond to the harmonic (1,1)-forms and (2,1)-forms. These forms are responsible for the

deformation of the "size" and the "shape" of this three-fold, respectively. In our case the exactly marginal operators act on causal structure. We proved that some causal spacetimes have counterparts in which causal protection is violated by CTC's. From the other side the causal structure is related to time but on the quantum mechanical side the operator corresponding to time does not exist [22]. Also in approach to quantization of gravity based on Wheeler-De Witt equation the time is vanishing as a consequence of the Hamiltonian constraint. We used the transformations from the conformal field theory, which is a starting point for quantization of the string theory, considered in this case as a perturbative quantum theory of gravitation. Thus these transformations represent the quantum effects related to the deformation of the causal structure of the target.

Here we only considered the Gödel-type metrics in (2+1) dimensions. We chose this example because it appears as the background for strings propagation. Thus we applied the transformations used by the conformal field theory. The second reason for considering this background is that it is (as the solution of equations of the general relativity) non-singular in the curvature tensor. Although this solution does not describe our universe it points out problems with the chronology in general relativity as it is well-known from long time.

## 5 Appendix

Relations between one-forms  $e$  and metric  $g$  are:

$$g_{\alpha\beta} = e_\alpha^a e_\beta^b \eta_{ab}, \quad (\text{A.1})$$

where Minkowski metric  $\eta_{ab} = \text{diag}(-1, 1, 1)$ . The measure  $\sqrt{-g}d^3x$  is expressed by  $e$  as follows:

$$\sqrt{-g}d^3x = \frac{1}{6}\varepsilon_{abc}e^a \wedge e^b \wedge e^c. \quad (\text{A.2})$$

The vierbains  $e^a$  are expressed by connections  $A_{L,R}$  as follows:  $e^a = (A_L^a - A_R^a) / (2l)$ .

The metric (2.13) has the four Killing vectors  $\xi_{(1)}, \dots, \xi_{(4)}$  which span the  $so(2) \times sl(2, \mathbf{R})$  algebra:

$$\begin{aligned} [\xi_{(1)}, \xi_{(r)}] &= 0, \\ [\xi_{(2)}, \xi_{(3)}] &= -\frac{1}{a}\xi_{(2)}, \quad [\xi_{(2)}, \xi_{(4)}] = +\frac{1}{a}\xi_{(3)}, \\ [\xi_{(3)}, \xi_{(4)}] &= -\frac{1}{a}\xi_{(4)}. \end{aligned} \quad (\text{A.3})$$

The concrete forms of those vectors depend on the choose coordinates in the metric. In the above notation the abelian part  $so(2)$  is spanned by  $\xi_{(1)}$ . As one can shown these metrics are solutions of the Einstein equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \Omega^2 g_{\alpha\beta} = T_{\alpha\beta}, \quad (\text{A.4})$$

with the energy-momentum tensor  $T_{\alpha\beta}$  is given by:

$$T_{\alpha\beta} = (4\Omega^2 - m^2) g_{\alpha 0} g_{\beta 0} \quad (\text{A.5})$$

and  $\alpha, \beta = 0, 1, 2$ . In the case when  $m^2 = 4\Omega^2$  the space-time becomes the AdS with the metric:

$$ds_{(AdS)}^2 = - \left( dt + \frac{1}{\Omega} \sinh^2(\Omega\rho) d\phi \right)^2 + d\rho^2 + \frac{1}{4\Omega^2} \sinh^2(2\Omega\rho) d\phi^2. \quad (\text{A.6})$$

In order to get standard form we have to make the following change of coordinates  $\sigma = \Omega t$ ,  $\psi = \Omega t - \phi$  and  $\tilde{\rho} = \Omega\rho$ . As the result one obtains:

$$ds_{(AdS)}^2 = \frac{1}{\Omega^2} \left( -\cosh^2(\tilde{\rho}) d\sigma^2 + d\tilde{\rho}^2 + \sinh^2(\tilde{\rho}) d\psi^2 \right). \quad (\text{A.7})$$

## 6 References

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